Inductive Reasoning

Bayes Rule

Die throw

1,2,3

4,5,6

A die throw determines from which urn to select balls. For outcomes 1, 2, and 3, balls are picked from urn A; for the other outcomes, balls are picked from urn B.

1) After the urn is selected by a die throw, but before any draw is seen, what is the probability that urn A is being used? _______________

2) The color of the first ball is black.
   a) What is the probability that A was selected? _____________
   b) What is the probability that B was selected? _____________

3) The first ball is put back into the urn and a second black ball is selected from the same urn. What is the probability that A is being used? ______

Some common answers

Incorrect:
2a) 1/2  b) 1/2
Each urn is equally likely to be selected beforehand but what did you learn from the draw?

2a) (1/2)(2/3)  (b) (1/2)(1/3)
probabilities have to sum up to one – the outcome is either urn A or B. Hint: need to rescale the numbers to make them add up to one.

Correct:
1) 1/2

2a) (1/2)(2/3) / [(1/2)(2/3)+(1/2)(1/3)] = 2/3
2b) (1/2)(1/3) / [(1/2)(2/3)+(1/2)(1/3)] = 1/3
Urn problem (2)

Now the problem is a little different....
Only when the die comes up with number “1”, balls are picked from urn A; for all other outcomes, balls are picked from urn B.

1) After the urn is selected by a die throw, but before any draw is seen, what is the probability that urn A is being used? ____________

2) The color of the first ball is black.
   a) What is the probability that A was selected? ____________
   b) What is the probability that B was selected? ____________

3) The first ball is put back into the urn and a second black ball is selected from the same urn. What is the probability that A is being used? _______

Die throw

1  2,3,4,5,6
A  B

Answers

Correct:
1) 1/6
2a) (1/6)(2/3) / [(1/6)(2/3)+(5/6)(1/3)] = 2/7
2b) (1/6)(1/3) / [(1/6)(2/3)+(5/6)(1/3)] = 5/7

Medical Test

• In the 1980’s in the US, a HIV test was used that had the following properties:
  There were 4% false positives
  There were 100% true positives

• About 0.4% of the male population was HIV positive

• If a man tested HIV positive, what is the probability he is actually HIV positive?
A counting heuristic

- Let’s take 1000 people.
- On average, 4 out of 1000 actually have the disease and all 4 of those will test positive (100% true positive rate)
- Among the 996 who do not have the disease, the test will falsely identify 4% as having it. 4% of 996 ≈ 40
- On average, out of 1000 people: 4 test positive and they have the disease 40 test positive and they do not have the disease.
- Therefore, a positive test outcome implies a 4/(4+40)≈9% chance of having the disease

Reasoning with Uncertainty

- Often, we want to reason from observable information to unobservable information
- We want to calculate how our prior beliefs change given new available evidence
- Bayes rule tells us how to optimally reason with uncertainty. Do people reason like Bayes rule?

Example of reasoning with uncertainty

- Medical diagnosis:
  - before seeing the patient, there is a prior probability that the patient has a disease. After diagnosis and running tests, the doctor assesses the posterior probability that the patient has a disease
  - test outcomes, symptoms (observable) → disease (unobservable)
- Law
  - (real story) Several elderly patients die. A nurse is accused of mishandling the patients under her care. Some lawyers argue that the elderly patients could have died by chance. How likely is it the nurse is guilty?
  - facts (observable) → guilt (unobservable, uncertain)
- Scientific reasoning:
  - Before running an experiment, scientists have a certain belief in their hypothesis being true. Their belief in their theory is updated in the light of the evidence gathered from experiments
  - evidence (observations from experiment) → hypothesis (unobservable)
Bayes Rule

Bayes rule tells us how the available evidence should alter our belief in something being true.

- Evidence: \( E_1 = \) outcome 1
  \( E_2 = \) outcome 2
- Hypotheses: \( H_1 = \) hypothesis 1
  \( H_2 = \) hypothesis 2
- Given:
  \( P(H_1) \) (prior probability)
  \( P(H_2) \) (prior probability)
  \( P(E_1 | H_1) \) (likelihood)
  \( P(E_1 | H_2) \) (likelihood)

- We can calculate: \( P(H_1 | E_1) \) (posterior prob.)

Bayes Rule ('flipping rule')

\[
P(H_1 | E_1) = \frac{P(E_1 | H_1)P(H_1)}{P(E_1 | H_1)P(H_1) + P(E_1 | H_2)P(H_2)}
\]

(the denominator normalizes the probabilities)
Medical Test

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Representation for medical problem

E₁ Positive Test Outcome
E₂ Negative Test Outcome
H₁ HIV positive
H₂ Not HIV positive

Prior probability: P(H₁) = .004, P(H₂) = .996

False positive: P(E₁ | H₂) = .04   P(E₂ | H₂) = .96
False negative: P(E₂ | H₁) = 0   P(E₁ | H₁) = 1

P(H₁ | E₁) = ??

Applying Bayes Rule

\[ P(H₁ | E) = \frac{P(E | H₁)P(H₁)}{P(E | H₁)P(H₁) + P(E | H₂)P(H₂)} \]
\[ = \frac{(1.0)(.004)}{(1.0)(.004) + (.04)(.996)} \]
\[ = .091 \]
Normative Model

- Bayes rule tells you how you should reason with probabilities – it is a prescriptive (i.e., normative) model.
- But do people reason like Bayes? (Tversky & Kahneman)
  - Bayes rate neglect
  - Conservatism

The Taxi Problem

- A witness sees a crime involving a taxi in Carborough. The witness says that the taxi is blue. It is known from previous research that witnesses are correct 80% of the time when making such statements.
- What is the probability that a blue taxi was involved in the crime?

The Taxi Problem

- A witness sees a crime involving a taxi in Carborough. The witness says that the taxi is blue. It is known from previous research that witnesses are correct 80% of the time when making such statements.
- The police also know that 15% of the taxis in Carborough are blue, the other 85% being green.
- What is the probability that a blue taxi was involved in the crime?
**Base Rate Neglect: The Taxi Problem**

- Failure to take prior probabilities (i.e., base rates) into account

- In the taxi story, the addition of:

  "The police also know that 15% of the taxis in Carborough are blue, the other 85% being green."

  has little influence on rated probability

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**Base Rate Neglect (2)**

- Kahneman & Tversky (1973). What is probability of picking an engineer in group A and B?

  Subject group A: 70 engineers and 30 lawyers
  Subject group B: 30 engineers and 70 lawyers

- Subjects can do this …

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**Providing additional information**

- "Jack is a 45 year-old man. He is married and has four children. He is generally conservative, careful, and ambitious. He shows no interest in political and social issues and spends most of his free time on his many hobbies, which include home carpentry, sailing, and mathematical puzzles”

- What now is probability Jack is an engineer?

- Both group A and group B gave $P = .9$
• “Dick is a 30-year-old man. He is married with no children. A man of high ability and high motivation, he promises to be quite successful in his field. He is well liked by his colleagues”

• What now is the probability that Dick is an engineer?

• Both group A and B gave P = .5

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**Conservatism**

Once people form a probability estimate, they are often slow to change the estimate given new information

URN A: 70 red balls, 30 blue balls
URN B: 30 red balls, 70 blue balls

A number of balls are selected from a randomly picked urn. What is probability of getting from urn A:

<table>
<thead>
<tr>
<th>Estimated Probability</th>
<th>Actual Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>One red</td>
<td>.60</td>
</tr>
<tr>
<td>Two red</td>
<td>.65</td>
</tr>
<tr>
<td>Three red</td>
<td>.70</td>
</tr>
</tbody>
</table>
